
IJMMS 2003:20, 1251–1260
 PII. S0161171203204269
<http://ijmms.hindawi.com>
 © Hindawi Publishing Corp.

MODULUS OF SMOOTHNESS AND THEOREMS CONCERNING APPROXIMATION ON COMPACT GROUPS

H. VAEZI and S. F. RZAEV

Received 3 April 2002

We consider the generalized shift operator defined by $(\text{Sh}_u f)(g) = \int_G f(tut^{-1}g)dt$ on a compact group G , and by using this operator, we define “spherical” modulus of smoothness. So, we prove Stechkin and Jackson-type theorems.

2000 Mathematics Subject Classification: 42C10, 43A77, 43A90.

1. Introduction. In this paper, we prove some theorems on absolutely convergent Fourier series in the metric space $L_2(G)$, where G is a compact group. The algebra of absolutely convergent Fourier series is a subject matter about which a good deal, although far from everything, is known (see [5, page 328]). Like many branches of harmonic analysis on T and R , the theory of absolutely convergent Fourier series is a fruitful source of questions about the corresponding entity for compact groups. By using some absolute convergence theorems of the classical Fourier series, (see [1, 11]), a generalized form of Stechkin [6] and Szasz theorem [1, 11] of the Fourier series on compact groups is obtained. Thus, we solve open problems formulated in [5, page 366] (see also [3, Chapter I, page 9]).

2. Preliminaries and notation. Now, we explain some of the notation and terminologies used throughout the paper.

Let G be a compact group with a dual space \hat{G} , $d\hat{g}$ denote the Haar measure on G normalized by the condition $\int_G d\hat{g} = 1$, and $\int_G f(\hat{g})d\hat{g}$ denote the Haar integral of a function f on G . Let U_α , $\alpha \in \hat{G}$ denotes the irreducible unitary representation of G in the finite dimensional Hilbert space V_α . We reserve the symbol d_α for the dimension of U_α . Thus, d_α is a positive integer. Also, we denote by χ_α and t_{ij}^α ($i, j = 1, 2, \dots, d_\alpha$), $\alpha \in \hat{G}$ the character and matrix elements (coordinate functions) of U_α , respectively.

Let $L_p(G)$ be the space of all functions f equipped with the norm

$$\|f\|_p = \left\{ \int_G |f(\hat{g})|^p d\hat{g} \right\}^{1/p}. \quad (2.1)$$

We write $\|\cdot\|_p$ instead of $\|\cdot\|_{L_p(G)}$, and $L_\infty = C$ is the corresponding space of continuous functions, and $\|f\| = \max\{|f(\hat{g})| : \hat{g} \in G\}$. As it is known (see [4])

or [10, page 99]), the space $L_2(G)$ can be decomposed into the sum

$$L_2(G) = \sum_{\alpha \in \hat{G}} \oplus H_\alpha, \quad (2.2)$$

where

$$H_\alpha = \{f \in C(G) : f(g) = \text{tr}(U_\alpha(g)C), C = \text{Hom}(V_\alpha, V_\alpha)\}. \quad (2.3)$$

This theorem is one of the most important results of the harmonic analysis on compact groups. The orthogonal projection $Y_\alpha : L_2(G) \rightarrow H_\alpha$ is given by

$$(Y_\alpha f)(g) = d_\alpha \int_G f(h) \chi_\alpha(gh^{-1}) dh, \quad (2.4)$$

where $(Y_\alpha f)(g)$ does not depend on the choice of a basis in L_2 . Carrying out this construction for every space H_α , $\alpha \in \hat{G}$, we obtain an orthonormal basis in L_2 consisting of the functions $\sqrt{d_\alpha} t_{ij}^\alpha$, $\alpha \in \hat{G}$, $1 \leq i, j \leq d_\alpha$. Any function $f \in L_2(G)$ can be expanded into a Fourier series with respect to this basis

$$f(g) = \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g), \quad (2.5)$$

where the Fourier coefficients a_{ij}^α are defined by the following relations:

$$a_{ij}^\alpha = d_\alpha \int_G f(g) \overline{t_{ij}^\alpha(g)} dg, \quad (2.6)$$

such that $\overline{t_{ij}^\alpha(g)} = t_{ij}^\alpha(g^{-1})$, where g^{-1} is the inverse of g . Note that (2.5) is a convergent series in the mean and that the Parseval's equality

$$\int_G |f(g)|^2 dg = \sum_{\alpha \in \hat{G}} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2 \quad (2.7)$$

holds. The aforementioned result of harmonic analysis on a compact group can be found, for example, in [4, 5, 7, 10].

We denote by Sh_u the generalized translation operator on compact group G defined by

$$\begin{aligned} (\text{Sh}_u f)(g) &= \int_G f(tut^{-1}g) dt, \\ (\triangle_u f)(g) &= f(g) - (\text{Sh}_u f)(g) = (E - \text{Sh}_u)f, \end{aligned} \quad (2.8)$$

where $u, g \in G$ and E is the identity operator. We set

$$\Delta_u^k f = \Delta_u \left(\Delta_u^{k-1} f \right) = (E - \text{Sh}_u)^k f = \sum_{i=0}^k (-1)^{k+i} C_k^i \text{Sh}_u^i f, \quad (2.9)$$

in which $\text{Sh}_u^0 f = f$ and $\text{Sh}_u(\text{Sh}_u^{i-1} f) = \text{Sh}_u^i f$, $i = 1, 2, \dots, k$ and $k \in \mathbb{N}$.

We note that α is a complicated index. Since \hat{G} is a countable set, there are only countably many $\alpha \in \hat{G}$ for which $\alpha_{ij}^\alpha \neq 0$ for some i and j ; enumerate them as $\{\alpha_0, \alpha_1, \dots, \alpha_n, \dots\}$. So, $d_{\alpha_0} < d_{\alpha_1} < d_{\alpha_2} < \dots < d_{\alpha_n} < \dots$. Because of that, the symbol “ $\alpha < n$ ” is interpreted as $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\} \subset \hat{G}$, and $\alpha \geq n$ denotes the set $\hat{G} \setminus (\alpha < n)$. Let d_α , as usual, be the dimension of U_α . For typographical convenience, we write d_n for the dimension of the representation U^{α_n} , $n = 1, 2, \dots$ (See [5, page 458].)

We denote by $E_n(f)_p$ the approximation of the function $f \in L_p(G)$ by “Spherical” polynomials of degree not greater than n :

$$E_n(f)_p = \inf \left\{ \|f - T_n\|_p : T_n \in \sum_{\alpha < n, \alpha \in \hat{G}} \oplus H_\alpha \right\}. \quad (2.10)$$

The sequence of best approximations $\{E_n(f)_p\}_{n=0}^\infty$ is a constructive characteristic of the function f . In the capacity of structural characteristic of the function f on a compact group G , we define its Spherical modulus of smoothness of order k by

$$\omega_k(f; \tau)_p = \sup \left\{ \|(E - \text{Sh}_u)^k f\|_p : u \in W_\tau \right\}, \quad (2.11)$$

where W_τ is a neighborhood of e in G . In other words,

$$W_\tau = \{u : \rho(u, e) < \tau, u \in G\}, \quad (2.12)$$

where ρ is a pseudometric on G and τ is any positive real number. It is easy to show the following properties of $\omega_k(f, \tau)_p$:

- (a) $\lim_{\tau \rightarrow 0} \omega_k(f, \tau)_p = 0$;
- (b) $\omega_k(f, \tau)_p$ is a continuous monotonically increasing function with respect to τ ;
- (c) $\omega_k(f_1 + f_2, \tau)_p \leq \omega_k(f_1, \tau)_p + \omega_k(f_2, \tau)_p$;
- (d) $\omega_{k+l}(f, \tau)_p \leq 2^l \omega_k(f, \tau)_p$, $l = 1, 2, \dots$.

3. Main results. We need the following simple but useful lemma.

LEMMA 3.1. *The following equality holds for all $u, g \in G$:*

$$(\text{Sh}_u t_{ij}^\alpha)(g) = \frac{\chi_\alpha(u)}{d_\alpha} t_{ij}^\alpha(g). \quad (3.1)$$

PROOF. Using the orthogonality relations and other formulas for matrix elements $t_{ij}^\alpha(g)$ (see [7, page 189]), we have

$$\begin{aligned} \int_G t_{ij}^\alpha(tut^{-1}g)dt &= \sum_{p=1}^{d_\alpha} \sum_{q=1}^{d_\alpha} t_{qp}^\alpha(u) t_{ij}^\alpha(g) \int_G t_{iq}^\alpha(t) \overline{t_{qp}^\alpha(t)} dt \\ &= \frac{1}{d_\alpha} \sum_{p=1}^{d_\alpha} t_{pp}^\alpha(u) t_{ij}^\alpha(g) = \frac{1}{d_\alpha} \chi_\alpha(u) t_{ij}^\alpha(g). \end{aligned} \quad (3.2)$$

This proves the lemma. \square

The following formula is the particular event of the above lemma:

$$\int_G \chi_\alpha(tut^{-1}g)dt = \frac{\chi_\alpha(u)\chi_\alpha(g)}{d_\alpha}. \quad (3.3)$$

It can be called a Weyl formula.

We note that the expansion (2.5) is connected with the expansion

$$f(g) = \sum_{\alpha \in G} Y_\alpha(f)(g), \quad Y_\alpha \in H_\alpha, \quad (3.4)$$

which is defined by (2.4), that is, by the equality

$$Y_\alpha(f)(g) = \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g). \quad (3.5)$$

Thus, the coefficients a_{ij}^α are defined by (2.6). Using Lemma 3.1 and the definition of Y_α , we obtain

$$\begin{aligned} Y_\alpha(\text{Sh}_u f)(g) &= \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha \int_G t_{ij}^\alpha(tut^{-1}g)dt \\ &= \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha \frac{\chi_\alpha(u)}{d_\alpha} t_{ij}^\alpha(g) \\ &= \frac{\chi_\alpha(u)}{d_\alpha} Y_\alpha(f)(g). \end{aligned} \quad (3.6)$$

The following are simple facts with frequent usage: if $f \in L_p$, then

- (1) $\|\text{Sh}_u f\|_p \leq \|f\|_p$;
- (2) $\|f - \text{Sh}_u f\|_p \rightarrow 0$ as $u \rightarrow e$;
- (3) $(Y_\alpha(\text{Sh}_u f))(g) = (\chi_\alpha(u)/\chi_\alpha(e))(Y_\alpha f)(g)$ for all $\alpha \in \hat{G}$.

We note that $\chi_\alpha(e) = d_\alpha$.

THEOREM 3.2. *If $f \in L_2$ and f is not constant, then*

$$E_n(f)_2 \leq \sqrt{\frac{d_n}{d_n - 2k}} \omega_k\left(f; \frac{1}{n}\right)_2, \quad n = 1, 2, \dots \quad (3.7)$$

PROOF. Let $f \in L_2$ and $S_n(f, g)$ denote the n th partial sum of the Fourier series (2.5), that is,

$$S_n(f, g) = \sum_{\alpha < n} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g) = \sum_{p=0}^n \sum_{i,j=1}^{d_{\alpha p}} a_{ij}^{\alpha p} t_{ij}^{\alpha p}(g). \quad (3.8)$$

Using Parseval's equality for the compact group G , we have

$$E_n^2(f)_2 = \|f - S_n(f)\|_2^2 = \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2. \quad (3.9)$$

Using (3), it is not hard to see that

$$(Y_\alpha(\Delta^k f))(g) = \left(1 - \frac{\chi_\alpha(u)}{d_\alpha}\right)^k (Y_\alpha f)(g), \quad \alpha \in \hat{G}. \quad (3.10)$$

Consequently, $(\Delta^k f)(g) = \sum_{\alpha \in \hat{G}} (1 - \chi_\alpha(u)/d_\alpha)^k a_{ij}^\alpha t_{ij}^\alpha$. By another application of Parseval's equality, we obtain

$$\begin{aligned} \|\Delta_u^k f\|_2^2 &= \sum_{\alpha \in \hat{G}} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left|1 - \frac{\chi_\alpha(u)}{d_\alpha}\right|^{2k} |a_{ij}^\alpha|^2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left|1 - \frac{\chi_\alpha(u)}{d_\alpha}\right|^{2k} |a_{ij}^\alpha|^2 \\ &= \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left(1 - \frac{2\operatorname{Re} \chi_\alpha(u)}{d_\alpha} + \frac{|\chi_\alpha(u)|^2}{d_\alpha^2}\right)^k |a_{ij}^\alpha|^2. \end{aligned} \quad (3.11)$$

Now, using Bernolli's inequality $(1+x)^k \geq 1+kx$ for $x \geq -1$, we obtain

$$\|\Delta_u^k f\|_2^2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left(1 - \frac{2k\operatorname{Re} \chi_\alpha(u)}{d_\alpha} + \frac{k|\chi_\alpha(u)|^2}{d_\alpha^2}\right) |a_{ij}^\alpha|^2. \quad (3.12)$$

Consequently,

$$\|\Delta_u^k f\|_2^2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2 - \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \frac{2k\operatorname{Re} \chi_\alpha(u)}{d_\alpha} |a_{ij}^\alpha|^2; \quad (3.13)$$

therefore,

$$E_n^2(f)_2 \leq \|\Delta_u^k f\|_2^2 + 2k \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \frac{\operatorname{Re} \chi_\alpha(u)}{d_\alpha} |a_{ij}^\alpha|^2. \quad (3.14)$$

Let Φ_{W_τ} be a nonnegative integrable function vanishing outside W_τ and satisfying the condition $\int_G \Phi_{W_\tau}(g) dg = 1$. For example, we can take $\Phi_{W_\tau} = \xi_{W_\tau} / \mu(W_\tau)$, where $\mu(W_\tau)$ is the Haar measure of W_τ and ξ_{W_τ} is the characteristic function of W_τ . Multiplying both sides of (3.14) by $\Phi_{W_{1/n}}$, and integrating with respect to u on G , and using the equality $\int_G |\chi_\alpha|^2 dg = 1$ (see [7, page 195]), we obtain

$$\begin{aligned} \int_G E_n^2(f)_2 \Phi_{W_{1/n}}(u) du &\leq \int_G \left\| \Delta_u^k f \right\|_2^2 \Phi_{W_{1/n}} du \\ &\quad + 2k \sum_{\alpha \geq n} \frac{1}{d_\alpha^2} \sum_{i,j=1}^{d_\alpha} \left| a_{ij}^\alpha \right|^2 \int_G |\chi_\alpha(u)| \Phi_{W_{1/n}}(u) du \quad (3.15) \\ &\leq \sup \left\| \Delta_u^k f \right\|_2^2 + \frac{2k}{d_n} \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left| a_{ij}^\alpha \right|^2. \end{aligned}$$

Therefore, it is not hard to see that

$$E_n^2(f)_2 \leq \omega_k^2 \left(f, \frac{1}{n} \right)_2 + \frac{2k}{d_n} E_n^2(f)_2. \quad (3.16)$$

Finally, we obtain

$$E_n(f)_2 \leq \sqrt{\frac{d_n}{d_n - 2k}} \omega_k \left(f, \frac{1}{n} \right)_2, \quad (3.17)$$

which proves the theorem. \square

This theorem is given without proof in [8] for the case where $k = 1$.

We note that the matrix elements of unitary representations $t_{ij}^\alpha(g)$ satisfy the relations

$$\sum_{j=1}^{d_\alpha} t_{ij}^\alpha(g) \overline{t_{kj}^\alpha(g)} = \sum_{j=1}^{d_\alpha} t_{ij}^\alpha(g) \overline{t_{jk}^\alpha(g)} = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases} \quad (3.18)$$

In particular, we have

$$\sum_{j=1}^{d_\alpha} \left| t_{ij}^\alpha \right|^2 = 1 \Rightarrow \left| t_{ij}^\alpha(g) \right| \leq 1 \quad (3.19)$$

for all $\alpha \in \hat{G}$ and $i, j = 1, 2, \dots, d_\alpha$. Furthermore, it is obvious that $|a_{ij}^\alpha t_{ij}^\alpha(g)| \leq |a_{ij}^\alpha|$; therefore, according to the sufficient condition for absolutely convergent Fourier series on the group G , the series $\sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|$ is convergent. Let $A(G) := \{f : \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha| < +\infty\}$. Using Theorem 3.2, and repeating the proof of analogous theorems (see [1, Chapter IX] or [6, Chapter II]) with some changes, we obtain the following theorems.

THEOREM 3.3. *If $f(g) \in L_2(G)$, then*

$$\sum_{n=1}^{\infty} \frac{\omega_k(f, 1/n)_2}{\sqrt{n}} < +\infty \Rightarrow f(g) \in A(G). \quad (3.20)$$

This theorem is analogous to the Szasz theorem of the classical Fourier series in the case where $k = 1$ and $G = T$.

THEOREM 3.4. *If $f(g) \in L_2(G)$, then*

$$\sum_{n=1}^{\infty} \frac{E_n(f)_2}{\sqrt{n}} < +\infty \Rightarrow f(g) \in A(G). \quad (3.21)$$

This theorem is also analogous to a theorem in trigonometric case proved by Stechkin [9].

4. Applications to compact group $SU(2)$. The group $SU(2)$ consists of unimodular unitary matrices of the second order, that is, matrices of the form

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (4.1)$$

Therefore, each element u of $SU(2)$ is uniquely determined by a pair of complex numbers α and β such that $|\alpha|^2 + |\beta|^2 = 1$. We have (see [5]) the relation “ $(\alpha, \beta) \mapsto (\phi, \theta, \psi)$,” where $\alpha\beta \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$, and the parameters ϕ , θ , and ψ are called Euler angles defined by

$$|\alpha| = \cos \frac{\theta}{2}; \quad \text{Arg} \alpha = \frac{\phi + \psi}{2}; \quad \text{Arg} \beta = \frac{\phi - \psi}{2}. \quad (4.2)$$

Let ϕ , θ , and ψ satisfy the conditions

$$0 \leq \phi < 2\pi, \quad 0 \leq \theta < \pi, \quad -2\pi \leq \psi < 2\pi. \quad (4.3)$$

Also, we know that the dimension of the representation T^l of $SU(2)$ is equal to $2l + 1$, where $l = 0, 1/2, 1, \dots$ and the matrix elements of T^l for group $SU(2)$ are defined by

$$t_{mn}^l(u) = e^{-(n\psi + m\phi)} P_{mn}^l(\cos \theta) i^{(m-n)}. \quad (4.4)$$

Expressing $t_{mn}^l(u)$ in terms of $P_{mn}^l(\cos \theta)$, we arrive at the following conclusion:

Any function $f(\phi, \theta, \psi)$, $0 \leq \phi < 2\pi$, $0 \leq \theta < \pi$, and $-2\pi \leq \psi < 2\pi$ belonging to the space $L^2(SU(2))$ such that

$$\int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi |f(\phi, \theta, \psi)|^2 \sin \theta d\theta d\phi d\psi < \infty \quad (4.5)$$

can be expanded into the mean-convergent series

$$f(\phi, \theta, \psi) = \sum_l \sum_{m=-l}^l \sum_{n=-l}^l \alpha_{mn}^l e^{-i(m\phi+n\psi)} P_{mn}^l(\cos \theta), \quad (4.6)$$

where

$$\alpha_{mn}^l = \frac{2l+1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi f(\phi, \theta, \psi) e^{i(m\phi+n\psi)} P_{mn}^l(\cos \theta) \sin \theta d\theta d\phi d\psi. \quad (4.7)$$

In addition, we obtain from Parseval's equality that

$$\sum_l \sum_{m=-l}^l \sum_{n=-l}^l \frac{1}{2l+1} |\alpha_{mn}^l|^2 = \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi |f(\phi, \theta, \psi)|^2 \sin \theta d\theta d\phi d\psi. \quad (4.8)$$

Using [Theorem 3.2](#), we obtain the following theorem.

THEOREM 4.1. *If $f(\phi, \theta, \psi) \in L_2(\text{SU}(2))$, then*

$$E_n(f)_2 \leq \sqrt{1 + \frac{2}{n-1}} \omega_k\left(f, \frac{1}{n}\right)_2, \quad (4.9)$$

$$\left\{ \sum_{l \geq n} \sum_{m=-l}^l \sum_{n=-l}^l \frac{1}{2l+1} |\alpha_{mn}^l|^2 \right\}^{1/2} \leq \sqrt{1 + \frac{2}{n-1}} \omega_k\left(f, \frac{1}{n}\right)_2.$$

Using the relation between the polynomial $P_n^{(\alpha, \beta)}(z)$ and $P_{mn}^l(z)$, we conclude that

$$P_{mn}^l(z) = 2^{-m} \left[\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!} \right]^{1/2} (1-z)^{(m-n)/2} (1+z)^{(m+n)/2} P_{l-m}^{(m-n, m+n)}. \quad (4.10)$$

The Jacobi polynomials obtained here are characterized by the condition that α and β are integers and $n + \alpha + \beta \in Z_+$.

Now, we consider the following case.

Let $L_2^{(\alpha, \beta)}[-1, 1]$ be the Hilbert space of the functions f defined on the segment $[-1, 1]$ with the scalar product

$$(f_1, f_2) = \int_{-1}^1 f_1(x) \overline{f_2(x)} (1-x)^\alpha (1+x)^\beta dx; \quad (4.11)$$

then, any function f in this space is expanded into the mean-convergent series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \hat{P}_n^{(\alpha, \beta)}(x), \quad (4.12)$$

where the polynomials $\hat{P}_n^{(\alpha, \beta)}(x)$ are given by

$$\hat{P}_k^{(\alpha, \beta)}(x) = 2^{-(\alpha+\beta+1)/2} \left[\frac{k!(k+\alpha+\beta)!(\alpha+\beta+2k+1)}{(k+\alpha)!(k+\beta)!} \right]^{1/2} P_k^{(\alpha, \beta)}(x), \quad (4.13)$$

$$\alpha_n = \int_{-1}^1 f(x) \hat{P}_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx. \quad (4.14)$$

The Parseval's equality

$$\int_{-1}^1 |f(x)|^2 (1-x)^\alpha (1+x)^\beta dx = \sum_{n=0}^{\infty} |\alpha|^2 \quad (4.15)$$

holds. The formulas (4.12), (4.14), and (4.15) are proved for integral nonnegative values of α and β . We can show that they are valid for arbitrary real values of α and β exceeding -1 . Finally, we reach the following theorem.

THEOREM 4.2. *If $f(x) \in L_2[-1, 1]$, then the following hold for Jacobi series:*

$$\begin{aligned} E_n(f)_2 &\leq \sqrt{1 + \frac{2}{n-1}} \omega_k\left(f, \frac{1}{n}\right)_2, \\ \left\{ \sum_{l=n}^{\infty} |\alpha_l|^2 \right\}^{1/2} &\leq \sqrt{1 + \frac{2}{n-1}} \omega_k\left(f, \frac{1}{n}\right)_2. \end{aligned} \quad (4.16)$$

NOTE. For the ideas similar to this paper we refer to [2] and its references.

ACKNOWLEDGMENTS. This research was supported by Tabriz University. We would like to thank the research office of Tabriz University for its support.

REFERENCES

- [1] W. K. Bari, *Trigonometric Series*, vol. II, Holt, Rinehart and Winston, New York, 1967.
- [2] G. Benke, *Bernšteĭn's theorem for compact groups*, J. Funct. Anal. **35** (1980), no. 3, 295–303.
- [3] R. E. Edwards, *Fourier series. A Modern Introduction. Vol. 1*, 2nd ed., Graduate Texts in Mathematics, vol. 64, Springer-Verlag, New York, 1979.
- [4] S. Helgason, *Groups and Geometric Analysis*, Pure and Applied Mathematics, vol. 113, Academic Press, Florida, 1984.
- [5] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups*, Die Grundlehren der Mathematischen Wissenschaften, vol. 152, Springer-Verlag, New York, 1970 (German).
- [6] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 50, Springer-Verlag, Berlin, 1970 (French).
- [7] M. A. Naimark and A. I. Štern, *Theory of Group Representations*, Grundlehren der Mathematischen Wissenschaften, vol. 246, Springer-Verlag, New York, 1982 (German).

- [8] S. F. Rzaev, *L_2 -Approximation on compact groups*, Proc. "Questions on Functional Analysis and Mathematical Physics Conference", Baku, 1999, pp. 418–419.
- [9] S. B. Stechkin, *On absolute convergence of orthogonal series*, Dokl. Akad. Nauk. SSSR **102** (1955), 37–40.
- [10] N. Ja. Vilenkin and A. U. Klimyk, *Representation of Lie groups and Special Functions. Vol. 1*, Mathematics and Its Applications, vol. 72, Kluwer Academic Publishers, Dordrecht, 1991.
- [11] A. Zygmund, *Trigonometric Series. 2nd ed. Vols. I, II*, Cambridge University Press, New York, 1959.

H. Vaezi: Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

E-mail address: hvaezi@tabrizu.ac.ir

S. F. Rzaev: Institute of Mathematics and Mechanics, Azerbaijan Academy of Sciences, Baku, Azerbaijan

E-mail address: rzseymur@hotmail.com

Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk